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# A remark on an oscillation constant in the half-linear oscillation theory

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ABSTRACT

We establish a new oscillation criterion for the half-linear second order differential equation

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x,\ p > 1,$$

which improves a result given in [8].



## 1. Introduction

In this paper we prove a new oscillation criterion for the half-linear second order differential equation

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x,\ p > 1, \tag{1}$$

where $r$, $c$ are continuous functions, $r(t) > 0$. In this criterion, equation (1) is viewed as a perturbation of the nonoscillatory equation of the same form

$$\big(r(t)\Phi(x')\big)' + \tilde{c}(t)\Phi(x) = 0, \quad p > 1, \tag{2}$$

and oscillation criterion for (1) is formulated in terms of the asymptotic behavior of the integral

$$\int^t [c(s) - \tilde{c}(s)]h^p(s)\,\mathrm{d}s,$$

where $h$ is a function "close" to the so-called nonprincipal solution of (2).

A typical example of this approach is the investigation of the equation

$$(\Phi(x'))' + c(t)\Phi(x) = 0 \tag{3}$$

as a perturbation of the half-linear Euler equation with the critical coefficient

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0, \quad \gamma_p := \left(\frac{p-1}{p}\right)^p, \tag{4}$$

see e.g. [1–7].

In the recent paper [8], the following oscillation criterion has been proved.

**Proposition 1.** *Let $\tilde{x}$ be the positive principal solution of (1) such that*

$$\liminf_{t\to\infty} |G(t)| > 0, \quad G(t) := r(t)\tilde{x}(t)\Phi(\tilde{x}'(t)), \tag{5}$$

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*and*

$$\int^{\infty} \frac{\mathrm{d}t}{R(t)} = \infty, \quad R(t) := r(t)\tilde{x}^2(t)|\tilde{x}'(t)|^{p-2}. \tag{6}$$

*If*

$$\liminf_{t\to\infty} \frac{1}{\int_T^t R^{-1}(s)\,\mathrm{d}s} \int_T^t [c(s) - \tilde{c}(s)]\tilde{x}^p(s) \left(\int_T^s R^{-1}(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s > \frac{2}{q} \tag{7}$$

*for sufficiently large $T$ and $q := \frac{p}{p-1}$ being the conjugate number to $p$, then equation (1) is oscillatory. Moreover, if $c(t) \geq \tilde{c}(t)$ for large $t$, $\liminf$ in (7) can be replaced by $\limsup$.*

In our paper we show that constant $\frac{2}{q}$ in (7) can be replaced by a four times better constant $\frac{1}{2q}$. It is known from the linear oscillation theory (i.e., for (1) with $p = 2$) that the application of the variational principle (which is used in the proof of Proposition 1) gives usually four times worse oscillation constant (usually under slightly less restrictive assumptions) than the application of the Riccati technique and its modifications; see [9]. In our paper we show that a similar phenomenon also appears in the half-linear oscillation theory.

## 2. Preliminaries

The linear Sturmian separation theorem extends verbatim to (1), so this equation can be classified as oscillatory or nonoscillatory similarly as in the linear case. For more details concerning essentials of the half-linear oscillation theory we refer to [10, Chap. 3], [11], or to [12].

In our result the so-called *principal* solution of (1) appears. Nonoscillation of (1) implies the existence of a solution of the Riccati type differential equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \quad q = \frac{p}{p-1} \tag{8}$$

(related to (1) by the substitution $w = r\Phi(x'/x)$) which is defined on some interval $[T, \infty)$. Among all solutions of (8) there exists the *minimal one* $\tilde{w}$, minimal in the sense that for any other solution $w$ of (8) we have $\tilde{w}(t) < w(t)$ for large $t$. The *principal solution* $\tilde{x}$ of (1) is then the solution which "generates" the minimal solution $\tilde{w}$ via the Riccati substitution $\tilde{w} = r\Phi(\tilde{x}'/\tilde{x})$, i.e., it is given by the formula

$$\tilde{x}(t) = C \exp\left\{\int^t r^{1-q}(s)\Phi^{-1}(\tilde{w}(s))\,\mathrm{d}s\right\},$$

where $\Phi^{-1}(x) = |x|^{q-2}x$ is the inverse function of $\Phi$ and $C$ is a real constant. The *nonprincipal solution* of (1) is any solution linearly independent of the principal solution. For details concerning the construction and the basic properties of principal and nonprincipal solutions of (1) we refer to [13,14].

We will need the following result which is a combination of Theorem 2 and Theorem 4 of [15].

**Proposition 2.** *Suppose that (2) possesses a positive principal solution $\tilde{x}$ such that (5) and (6) hold. Further suppose that*

$$\int^{\infty} r^{1-q}(t)\,\mathrm{d}t = \infty, \tag{9}$$

*the below given integral in (10) is convergent, and*

$$\int_t^{\infty} \left[\tilde{c}(s) + \frac{1}{2q\tilde{x}^p(s)R(s)(\int_T^s R^{-1}(\tau)\,\mathrm{d}\tau)^2}\right] \mathrm{d}s > 0 \tag{10}$$

*for some $T \in \mathbb{R}$ and large $t$. If $g(t) \geq 0$ for large $t$ and*

$$\int^{\infty} g(t)\tilde{x}^p(t) \left(\int_T^t R^{-1}(s)\,\mathrm{d}s\right) \mathrm{d}t = \infty, \tag{11}$$

*then the equation*

$$(r(t)\Phi(x'))' + \left[\tilde{c}(t) + \frac{1}{2q\tilde{x}^p(t)R(t)(\int_T^t R^{-1}(s)\,\mathrm{d}s)^2} + g(t)\right]\Phi(x) = 0 \tag{12}$$

*is oscillatory.*

## 3. Main result

In this section we present the main result of the paper. We formulate an improvement of the oscillation criterion given in Proposition 1.

**Theorem 1.** *Let $\tilde{x}$ be the positive principal solution of (2) such that (5) and (6) hold. Further suppose that*

$$c(t) \geq \tilde{c}(t) + \frac{1}{2q\tilde{x}^p(t)R(t)(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s)^2} \tag{13}$$

*for some $T_0 \in \mathbb{R}$ and large $t$, and that (9) and (10) hold for large $t$. If*

$$\liminf_{t\to\infty} \frac{1}{\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s} \int_T^t [c(s) - \tilde{c}(s)]\tilde{x}^p(s) \left(\int_T^s R^{-1}(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s > \frac{1}{2q} \tag{14}$$

*for $T$ sufficiently large, then equation (1) is oscillatory.*

**Proof.** We rewrite equation (1) into the form

$$\left(r(t)\Phi(x')\right)' + \left[\tilde{c}(t) + \frac{1}{2q\tilde{x}^p(t)R(t)(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s)^2} + g(t)\right]\Phi(x) = 0,$$

where $g(t) = \left(c(t) - \tilde{c}(t)\right) - \frac{1}{2q\tilde{x}^p(t)R(t)(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s)^2} \geq 0$ for large $t$. According to Proposition 2, to prove oscillation of (1), it suffices to show that

$$\int^\infty \left[c(t) - \tilde{c}(t) - \frac{1}{2q\tilde{x}^p(t)R(t)(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s)^2}\right]\tilde{x}^p(t)\left(\int_T^t R^{-1}(s)\,\mathrm{d}s\right)\mathrm{d}t = \infty.$$

By (14), there exists $\varepsilon > 0$ such that

$$\frac{1}{\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s} \int_T^t \left[c(s) - \tilde{c}(s)\right]\tilde{x}^p(s)\left(\int_T^s R^{-1}(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s > \frac{1}{2q} + \varepsilon$$

for $t$ sufficiently large, say $t > \tilde{T}$. It means that

$$\frac{1}{R(t)} \int_T^t \left[c(s) - \tilde{c}(s)\right]\tilde{x}^p(s)\left(\int_T^s R^{-1}(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s > \frac{\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s}{R(t)}\left(\frac{1}{2q} + \varepsilon\right) \tag{15}$$

for $t > \tilde{T}$. For $b > \tilde{T}$, using integration by parts and (15) we have

$$\begin{aligned}
&\int_T^b \left[c(t) - \tilde{c}(t) - \frac{1}{2q\tilde{x}^p(t)R(t)(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s)^2}\right]\tilde{x}^p(t)\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)\mathrm{d}t \\
&\quad = \int_T^b \left[c(t) - \tilde{c}(t)\right]\tilde{x}^p(t)\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)\mathrm{d}t - \frac{1}{2q}\int_T^b \frac{\mathrm{d}t}{R(t)\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)} \\
&\quad = \int_T^b \left[c(t) - \tilde{c}(t)\right]\tilde{x}^p(t)\frac{\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)^2}{\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s}\mathrm{d}t - \frac{1}{2q}\log\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)\Bigg|_T^b \\
&\quad = \frac{1}{\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s}\int_T^t \left[c(s) - \tilde{c}(s)\right]\tilde{x}^p(s)\left(\int_{T_0}^s R^{-1}(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s\Bigg|_T^b \\
&\qquad + \int_T^b \frac{\int_T^t \left[c(s) - \tilde{c}(s)\right]\tilde{x}^p(s)\left(\int_{T_0}^s R^{-1}(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s}{\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)^2 R(t)}\mathrm{d}t - \frac{1}{2q}\log\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)\Bigg|_T^b \\
&\quad = \frac{1}{\int_T^b R^{-1}(t)\,\mathrm{d}t}\int_T^b \left[c(t) - \tilde{c}(t)\right]\tilde{x}^p(t)\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)^2 \mathrm{d}t + \int_T^{\tilde{T}} \frac{\int_T^t \left[c(s) - \tilde{c}(s)\right]\tilde{x}^p(s)\left(\int_{T_0}^s R^{-1}(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s}{\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)^2 R(t)}\mathrm{d}t \\
&\qquad + \int_{\tilde{T}}^b \frac{\int_T^t \left[c(s) - \tilde{c}(s)\right]\tilde{x}^p(s)\left(\int_{T_0}^s R^{-1}(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s}{\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)^2 R(t)}\mathrm{d}t - \frac{1}{2q}\log\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)\Bigg|_T^b
\end{aligned}$$

$$
\begin{aligned}
&> \frac{1}{2q} + \varepsilon + K + \left(\frac{1}{2q} + \varepsilon\right) \int_T^b \frac{1}{R(t)\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)}\,\mathrm{d}t - \frac{1}{2q} \log\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)\Bigg|_T^b \\
&= \frac{1}{2q} + \varepsilon + K + \left(\frac{1}{2q} + \varepsilon\right) \log\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)\Bigg|_T^b - \frac{1}{2q} \log\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)\Bigg|_T^b \\
&= \frac{1}{2q} + \varepsilon + K + \varepsilon \log\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)\Bigg|_T^b \longrightarrow \infty \quad \text{as } b \to \infty,
\end{aligned}
$$

where $K = \int_T^{\tilde{T}} \frac{\int_T^t [c(s)-\tilde{c}(s)]\tilde{x}^p(s)\left(\int_{T_0}^s R^{-1}(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s}{\left(\int_{T_0}^t R^{-1}(s)\,\mathrm{d}s\right)^2 R(t)}\,\mathrm{d}t$. □

**Remark 1.** (i) When we apply the previous theorem to the half-linear Riemann–Weber differential equation

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t}\right]\Phi(x) = 0, \quad \gamma_p = \left(\frac{p-1}{p}\right)^p, \tag{16}$$

regarded as a perturbation of (4) with $\tilde{x}(t) = t^{\frac{p-1}{p}}$ and $R(t) = \left(\frac{p-1}{p}\right)^{p-2} t$, we find the known result (see [2]) that (16) is oscillatory if $\mu > \mu_p := \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$.

(ii) In our main result we suppose (5) and (6). Condition (6) is closely related to the integral characterization of the principal solution of half-linear differential equations, we refer to [16,17] for discussion concerning this assumption. Condition (5) is technical, we needed it to prove an asymptotic formula for nonprincipal solution of (2); see [8]. The subject of the present investigation is to find a similar asymptotic formula in case when $\lim_{t\to\infty} G(t) = 0$.

(iii) Constant $\frac{1}{2q}$ in condition (14) is in good agreement with the nonoscillation criterion given in [1, Theorem 2] where it is shown (substituting $h = \tilde{x}(\int^t R^{-1})^{\frac{2}{p}}$ there) that (1) is nonoscillatory provided lim sup of the expression in (14) is less than $\frac{1}{2q}$. Also, restriction (13) is natural in view of [15, Theorems 1,2] which state that (1) is nonoscillatory provided $c(t) \le \tilde{c}(t) + \frac{1}{2q\tilde{x}^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}$.

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